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Der Pharmacia Lettre, 2021, 13 (7): 132-139
(http: //scholarsresearchlibrary. com/archive. html)

ISSN 0975-5071
USA CODEN: DPLEB4

A New Approach on the Finite Fractal Dimension of Some Chaotic Dynamics

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ABSTRACT

Dynamical systems, chaotic attractors and fractal dimension are mathematical concepts that have been intensively applied in much area of biomedical sciences. Recent studies on chaotic dynamics from Neuro-science and epidemiology lead to the existence of chaotic attractors and the question arises on their finite fractal dimensionality. Fractal dimensionality is accepted as a measure of complexity for systems that cannot be described by integer dimensions. A new idea, detailed in is presented in this review article that allows us to bind from above the fractal dimension of some chaotic attractors.

Keywords: Chaotic attractors, Dynamics, Fractal geometry, Fractal dimension, Capillary networks.

INTRODUCTION

Fractal geometry has become very useful in the understanding of many phenomena in various fields such as quantum physics, economy, and biology and recently in medicine. A common characteristic of inhomogeneous structures is the spatial or spatiotemporal self-similarity, which means that small-scale structures of the fractal set resemble large-scale structures.

The human body is full of a multitude of very complex structures; this is the case of the respiratory tract and the prodigious bronchial ramification, some parts of the heart, the renal system, the large blood and capillary networks, etc. (Figures 1 and 2).

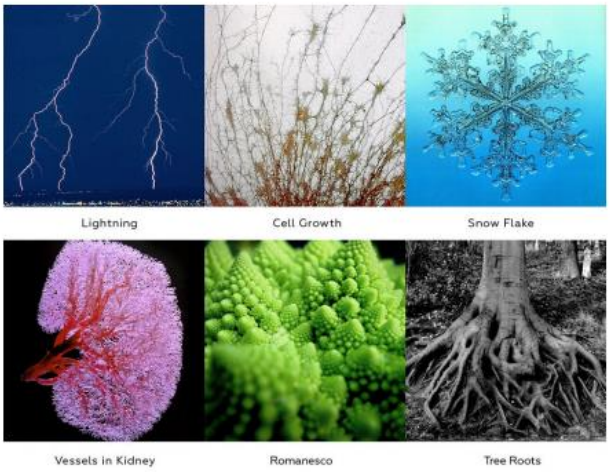

Figure 1: Complex structures.

Figure 2: Fractals in nature.

The study of fractals opened up a whole new world in regard to the study of the human body. Fractals are everywhere inside of us. Kidneys, blood vessels, lungs, even our brain exhibit fractal formations.

In order to emphasize the applications of fractal geometry and its contribution in biology and nuclear medicine, we note for example that fractal geometry is a key tool in the detection of cancerous cells *in vitro* and has provided the answer to the unusually long incubation period of the AIDS virus [1].

One of the important applications of fractals and their dimensions in the medical field, the diabetic retinopathy [2]. Fractal geometry and fractal dimension were used for the detection of early retinal vascular diseases [2] (Figure 3).

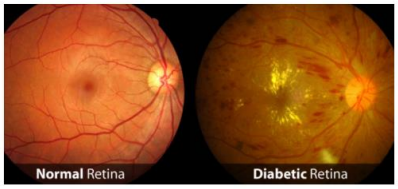

Figure 3: Fractals and their dimensions in the medical field.

FRACTAL ANALYSIS

Fractal analysis is a contemporary method of applying nontraditional mathematics to patterns that defy understanding with traditional Euclidean concepts. In essence, it measures complexity using the fractal dimension.

Identifying the presence of self-similarity properties can be useful in preliminary clinical trials for the diagnosis of cancerous epithelial diseases, blood and liver on the initial stage the analysis of digital images [3] as well as in the study of pathologies of the central nervous system. The way that a virus binds to a living molecule relies on the fractal structuring of both components. One of the biggest advantages of the fractal analysis is that it makes possible to estimate parameters of the objects, which are very complex and so complicated, that their mathematical description in terms of Euclidean geometry is impossible.

However, fractal analysis must be applied with certain caution in natural objects such as bio-medical ones.

The quantitative measure of the self-similar structure is the fractal dimension, which allows one to study the structure of objects and the relationship between it and the processes occurring in such complex systems as in bone tissues and the human brain. For the contribution of the fractal dimension to biology and medicine [1-6]. Fractal dimension or the box counting dimension is a statistical measure that correlates the morphological structural complexity of cellular components and biological tissues. Most biological elements, whether at cellular, tissue, or organ level, have self-similar structures that can be characterized by means of the fractal dimension.

From mathematical point of view, the fractal dimension or “box-counting dimension” is defined as follows [5,6].

Definition

The fractal dimension of a compact subsetM of a metric space H is defined by

$$D_f(M) = \lim_{\varepsilon \to 0} \frac{1_n \ N \ (\mathrm{M}, \varepsilon)}{1_n \left(\frac{1}{\varepsilon} \right)}$$

where N (M, ε) denotes the minimal number of closed balls of the radius s which cover the set M.

For simple issues on the basics of fractal analysis method known as the “box-counting” of the fractal dimension [7-9] (Figure 4).

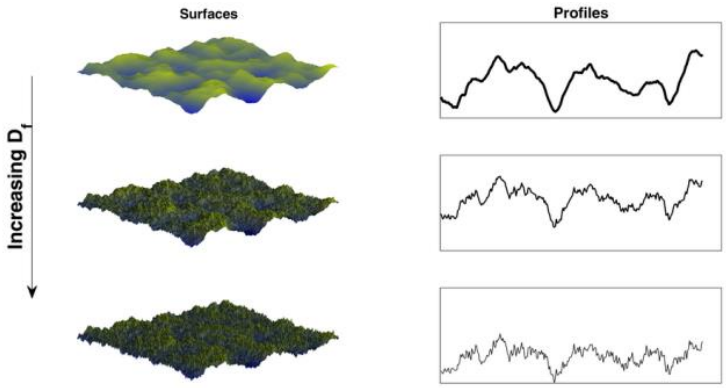

Figure 4: Fractal dimension.

CHAOTIC DYNAMICS

The chaotic dynamics is inherent not only to the continuous dynamic systems but to the discrete ones as well [7]. The small external fluctuations may cause the unpredictable behavior of the system as a whole. Catastrophical behavior in complex systems can occur not only due to external reasons, but also due to the fact that small events inside a system can add up together, lead to a chain reaction and spontaneous evolution to a critical state (cascade of bifurcations, turbulent). In the phase space of dissipative systems (the set of all possible states of a dynamic system), there appear the attracting sets with complex structure, a strange chaotic (or global) attractor that do not have a rigid periodic dynamics and possess a dimension different from the topological and therefore identified with fractals [6,7,10,11].

Biological systems manifest many of the features of chaotic systems including fractal structure and the existence of strange chaotic attractors as given by a parsimonious dynamical model for structural learning in the human brain [12,13] (Figure 5).

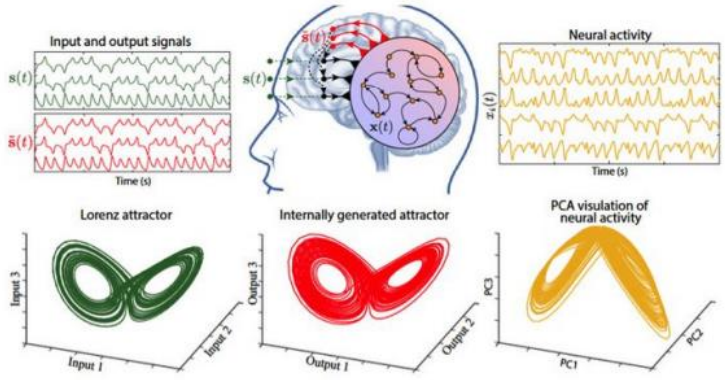

Figure 5: Parsimonious dynamical model.

Cancers are presented as complex systems exhibiting chaotic gene expression dynamics [14, 15]. Tumors are considered as chaotic attractors [16, 17]. The rich dynamic structure of the neuron firing patterns [18] as well as the chaotic evolution

of epidemics [19] has led to their being modeled by nonlinear dynamical systems [5].

Self-similarity in the dynamic structure of heart rates, as seen on a heart rate *vs.* time plots, reveals evidence of fractal dimension embedded in the behavior of the cardiac system [20]. By explicitly modeling the impulses of the human heart, we are able to apply chaotic attractor control techniques to modern pacemakers, and attempt to reduce the risk of heart attacks and other potentially life threatening cardiovascular problems [21].

THE MAIN IDEA

In the current review article, we present briefly a new approach [22]. Thought two basic models of nonlinear Schrödinger type equations, to bind from above the fractal dimension of some chaotic global attractors.

We often turn to apply Theorem which its main advantage is that we do not involve any smoothness properties for evolutions except the Lipschitz continuity. The basic idea behind this criterion is related to a squeezing property and involving a compactness criterion evaluated on trajectories. The compact part is described by means of compact semi-norms [10].

Theorem

"Let H be a Hilbert space and M be a bounded closed set in H. Assume that there exists a mapping $V: M \to H$ such that

1. $M \subseteq V(M)$. *Moreover,* V *is Lipschitz on* M, i.e, *ther exists* $L > 0$ *such that for all* $u_1, u_2 \in M, \left\|V(u_1) - V(u_2)\right\|_H \leq L\left\|u_1 - u_2\right\|_H$.

2. *There exist compact semi – norm* N *on* H *such that* $\forall\ u_1,\ u_2 \in M$,

$$\left\|V(u_1) - V(u)\right\|_H \leq \delta \left\|u_1 - u_2\right\|_H + K\, N(u_1 - u_2).$$

where $0 < \delta < 1$ *and* $K > 0$ *are constants.*

A semi norm N *on* H *is said to be compact if and only if every bounded sequence* $(x_n)_n$ *in* H, *possesses a subsequence* $(x_{nk})_k$ *satisfying* $N(x_{nk} - x_{nl}) \to 0$ *as* $k,\ l \to +\infty$.

Then M *is a compact set in* H *of a finite fractal dimension.*

Moreover, the following estimate holds

$$D_f(M) \leq \left[ln \frac{2}{1+\delta} \right]^{-1} ln \left[m_0 \left(\frac{4K(1+L^2)^{\frac{1}{2}}}{1-\delta} \right) \right],$$

Where $m_0(\mathbb{R})$ is the maximal number of $\chi_i \in H$ satisfying

$\| \chi_i \|_H \leq \mathbb{R}$ and $N(\chi_i - \chi_j) > 1$, $i \neq j$

Roughly speaking, the assumption means that the mapping V squeezes the set M along the space *H* up to modulo compact semi-norm, although it does not stretch M too much along *H*. This property is expressed by the assertion on finite dimensionality of M. Since we deal, more often than not, with infinite dimensional dynamical systems (i.e with infinite number of parameters) in unbounded domains, this compactness criterion is typical and cannot be guaranteed.

With the aims of explaining our new approach, it is advantageous for the convenience of the reader and for the sake of clarity to use a prototypal simple example, that is the one dimensional nonlinear Schrödinger equation (NLS), in order to provide the basic outline and the key steps of our strategy that is applied and discussed in further for different basic models [22-25].

Consider the one dimensional cubic dissipative NLS equation that reads

$u_t + iu_{xx} + i\,|u|^2 u + \gamma u = f,\ f \in L^2(\mathbb{R})$

Supplemented with the initial condition

$u(t=0) = u_0 \in H^1(\mathbb{R})$

Where $L^2(\mathbb{R})$ stands for the usual Lebesgue space of square integrable functions, H1 (R) the sobolev space and f an external source that is independent of time and belongs to $L^2(\mathbb{R})$ [26].

It is well known that the equation defines an infinite dimensional dissipative dynamical system that possesses a compact global attractor A in $H^1(\mathbb{R})$ [27, 28]. Moreover, if we assume that $xf(x) \in L^2(\mathbb{R})$ then the fractal dimension of A is finite. Similar results as the previous theorem can be found for instance in for other models of NLS type equations [29-31].

Our objective is to remove the additional condition that is $\gamma f(\gamma) \in L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ to ensure that the fractal dimension of the corresponding global attractor will always be finite without any further assumptions on the forcing term f. To do this, one only has to fulfill the conditions required in the Theorem 2.1 that mainly rely on exhibiting a compact semi-norm N on H1 (R) such that for a fixed time $t^* > 0$, the trajectories issued respectively from u_0 and v_0 satisfy.

$$\left\| u(t^*) - v(t^*) \right\|_{H^1(\mathbb{R})} \leq \delta \left\| u_0 - v_0 \right\|_{H^1(\mathbb{R})} + C\, N(u_0 - v_0)$$

for all $u_0, v_0 \in A$ *with* $0 < \delta < 1$.

Our idea consists in choosing, for the compact semi – norm, the L^2 *– restriction norm defined by*

$$N(u) = \|u\|_{L^2([-R,R])} = \left(\int_{|x| \leq R} |u(x)|^2 \, dx \right)^{\frac{1}{2}}$$

For a suitable choice of R > 0. In order to fulfill the condition of the theorem, a key estimate should be introduced and states as follows

Lemma: Let $p \in [1, +\infty)$ and C be a compact subset of $L^p(\mathbb{R})$.

Then for every $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that for all $\phi \in C$

$$\int_{\mathbb{R}} |\phi| |u|^2 \, dx \leq R \, \|u\|^2_{L^2(-R,R)} + \varepsilon \, \|u\|^2_{H^1(\mathbb{R})}, \quad \forall\, u \in H^1(\mathbb{R})$$

Using essentially the key Lemma, the regularity issue of the global attractor A and some necessary a priori estimates, for which we reader the reader to, we succeed to establish that [22,27].

$$\left\| u\left(t^{*}\right) - v\left(t^{*}\right) \right\|_{H^1(\mathbb{R})} \leq \frac{1}{2} \left\| u_0 - v_0 \right\|_{H^1(\mathbb{R})} + C \left\| u_0 - v_0 \right\|_{L^2([-R, R])}.$$

For suitable choice of $R > 0$ and $t^{*} > 0$.

CONCLUSION

Throughout our world, the intrinsic value of natural biological processes can be seen through the chaotic dynamical systems theory. From the fractal patterns in the naked branches of trees to the orbiting asteroids in outer space, chaotic attractors play a role in the character of life. With its generality, we believe that this framework serves as a promising preliminary tool with which more biomedical chaotic dynamical systems could be studied in the future to prompt better knowledge on their behavior and that allows us to gain a deeper understanding and information on their dynamic.

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